MOND effects in the inner solar system

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ABSTRACT

I pinpoint a previously unrecognized MOND effect that may act in the inner solar system, and is due to the galactic acceleration, $g_g = \eta a_0$: a byproduct of the MOND external-field effect. Predictions of the effect are not generic to the MOND paradigm, but depend on the particular MOND formulation at hand. However, the modified-Poisson formulation, on which I concentrate, uniquely predicts a subtle anomaly that may be detected in planetary and spacecraft motions (and perhaps in other precision systems, such as binary pulsars), despite their very high accelerations, and even if the MOND interpolating function is arbitrarily close to unity at high accelerations. Near the sun, this anomaly appears as a quadrupole field, with the acceleration at position $\mathbf{u}$ from the sun being $g^a_i(\mathbf{u}) = -q^a_{ij}u^j$, with q^a_{ij} diagonal, axisymmetric, and traceless: $-2q^a_{xx} = -2q^a_{yy} = q^a_{zz} = q(\eta)(a_0/R_M)$, where $R_M = (M_\odot G/a_0)^{1/2} \approx 8 \cdot 10^3$a.u. is the MOND radius of the sun. The anomaly is described and analyzed as the Newtonian field of the fictitious cloud of "phantom matter" that hovers around the sun. I find, for the relevant range of η values, and for a range of interpolating functions, $\mu(x)$, values of $10^{-2} \lesssim -q \lesssim 0.3$, which turn out to be sensitive to the form of $\mu(x)$ around the MOND-to-Newtonian transition. This range verges on the present bounds from solar system measurements. There might thus exist favorable prospects for either measuring the effect, or constraining the theory and the relevant parameters. Probing this anomaly may also help distinguish between modified inertia and modified gravity formulations of MOND. I also discuss briefly an anomaly that is generic to MOND in all its formulations, and competes with the quadrupole anomaly in the special case that $1 - \mu(x)$ vanishes as $x^{-3/2}$ as $x \to \infty$.

Subject headings:

1. Introduction–the MOND external-field effect

The MOND external-field effect (EFE) encapsulates the fact that even a constant-acceleration gravitational field in which a system is falling may enter and affect the internal dynamics of this system. It results from the non-linearity of MOND, which, in turn, follows from the basic premises of the paradigm. This means that even though all the constituents of the system fall in the same external field, this field does not cancel in their relative motion, as happens in (the linear) Newtonian

dynamics. The effect appears already in the pristine formulation of MOND (Milgrom 1983), as well as in the formulation of MOND as a modified gravity described by a nonlinear extension of the Poisson equation (Bekenstein & Milgrom 1984, Milgrom 1986a). Many applications to astrophysical systems have been discussed to date, including dwarf spheroidal galaxies in the field of a mother galaxy (Brada & Milgrom 2000a, Angus 2008), globular clusters (Haghi et al. 2009), warp induction by a companion (Brada & Milgrom 2000b), escape speed from a galaxy (Famaey, Bruneton, & Zhao 2007, Wu et al. 2007), binary galaxies (Tiret et al. 2007), departure from asymptotic flatness of the rotation curve (Wu et al. 2007), galaxy clusters (Angus & McGaugh 2008), and others.

In all these applications one deals with accelerations, both internal ($\mathbf{g}_{in}$) and external ($\mathbf{g}_{ex}$), that are of order of, or smaller than, the MOND acceleration constant a_0, and with external accelerations that are similar or larger than the internal ones. In these cases the pristine formulation and the modified-Poisson equation give similar results, barring order-of-unity differences in the geometry. In modified-inertia formulations of MOND the EFE still acts, but its implications can be quite different (Milgrom, in preparation).

Here I consider a situation in which the relative magnitudes of the three accelerations are very different: $g_{in} \gg g_g > a_0$ (in what follows I use everywhere $\mathbf{g}_{ex} = \mathbf{g}_g$ for the galactic field). This characterizes, for example, high acceleration systems in the galactic neighborhood of the sun, such as the inner solar system (SS) or binary stars. These three inequalities would each, separately, reduce the effect, and render it small. However, the effect might still be detectable in systems that are measured with high accuracy such as motions in the solar system, or in binary pulsars. This arrangement of accelerations does not lend itself to simple analytic, or numerical, considerations, as do the cases studied so far.

Small, nonrelativistic MOND anomalies in high-acceleration motions may possibly have several origins. In non-local, modified inertia formulations of MOND, anomalies may appear in the inner solar system in the motion of bodies that are on highly eccentric trajectories; trajectories that take them to large distances, where accelerations are low (Milgrom, in preparation). Cases in point are the long-period comets, and the Pioneer spacecraft. Because of the non-locality of such theories, the motions in high-acceleration portions of the orbit may be affected by the low acceleration ones, giving rise to small anomalies. Such MOND effects have been proposed as a possible mechanism for generating the Pioneer anomaly, without affecting the motions of planets, whose orbits are wholly in the high acceleration regime (Milgrom 2002, 2005, in preparation).

I concentrate on nonrelativistic, modified-gravity formulations, in which MOND is described by a modified potential field, in which particles move according to the standard laws of motion. In this case, an anomaly can arise in high acceleration systems due to a remaining departure of

the MOND interpolating function, $\mu(x)$, from unity, even at high accelerations.[1,2] For example, in the inner SS such a departure appears as an anomalous, spherical acceleration field. Its effects on planetary motions, and constraints on the high-x behavior of $\delta\mu(x) \equiv 1-\mu(x)$, have been discussed, e.g., in Milgrom (1983), and in Sereno & Jetzer (2006). In such analyses the sun is treated as an isolated point mass.

Here, I discuss another type of anomaly having to do with possible influence of the galactic field through the EFE. This, as we shall see, appears in the inner SS as an anomalous quadrupole field, not as a spherical one.

In order to isolate this effect from the first type, I shall use, all along, forms of μ with $\delta\mu(x) \to 0$ fast enough to dwarf the first effect in comparison with the EFE.

It is worth noting first that for the parameter values relevant to the SS, there is no unique MOND prediction of the EFE. The primitive, pristine formulation (Milgrom 1983), in which the MOND acceleration, $\mathbf{g}$, is calculated from the local Newtonian value, $\mathbf{g}_N$, via an algebraic relation of the form

$$\mathbf{g} = \nu(|\mathbf{g}_N|/a_0)\mathbf{g}_N \tag{1}$$

[where $y\nu(y) = x$ is the inverse of $x\mu(x) = y$], captures the main MOND effects in many instances, for example, in describing galaxy rotation curves, or the EFE when $g_g \gg g_{in}$ (Milgrom 1983). However, it is quite inapplicable to the case $g_g \ll g_{in}$, such as in solar system, as it is known not to give correctly the motion of such compound systems in an external field: For example, atoms in a star, or the stars in a close binary, moving in the outskirts of a galaxy are subject to high total accelerations because of the internal forces in the system. The naive formulation will say then that the full motion of these constituents is Newtonian. But this would imply that the whole star (or binary) is also moving Newtonically in the galaxy, contrary to what is observed, and to what we want MOND to accomplish. This is also the situation for the solar system, as the acceleration of the Sun due to Jupiter alone is $\sim 2500a_0$, and that of all the planets due to the sun, even larger. Applying the algebraic formulation to the motions of the Sun and planets, separately, would give not only the wrong motion for the solar system in the Galaxy, but also a negligible EFE for a fast decreasing $\delta\mu(x)$.

Modified-inertia formulations of MOND are more versatile and it is not possible to deduce a generic modified inertia prediction of the EFE (Milgrom, in preparation). For example, such formulations can be constructed in which there is no EFE in the inner SS. This is the case, e.g., in the

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[1]Galactic dynamics probe this function only at low values of $x \lesssim 5$, and gives no constraint whatever on the high x behavior. It is easy to invent forms of $\mu(x)$ with slow rise near $x = 1$, but very fast convergence to 1 at high x (see some examples in Milgrom & Sanders 2008).

[2]Anomalies may also appear in relativistic theories such as TeVeS and the like, because these theories do not exactly coincide with general relativity even in the limit $a_0 \to 0$ (see Bekenstein 2006, Sanders 2006, 2007, Sagi 2009, and Skordis 2009.)

toy theory described in Milgrom (1999). (This theory does exhibit an EFE in other circumstances.) But we do not yet have an acceptable formulation of MOND as modified inertia.

However, it turns out that in the modified Poisson formulation of Bekenstein & Milgrom (1984) there is an EFE that is felt in the inner SS as a quadrupole anomaly (to dominant order). This anomaly exists despite the high accelerations of the planets, and, interestingly, it remains finite even with $\delta\mu(x) \rightarrow 0$ arbitrarily fast at high x. Furthermore, the strength of the effect is sensitive to the form of $\mu(x)$ in the MOND-to-Newtonian transition region $x \sim 1$, even though the effect itself is probed at very high accelerations.[3]

All this results from properties of the Poisson equation, and its ilk, such as the non-linear version treated here: Distortion of the field in one place is felt in all other regions. Here, the distortion is created in the sub a_0 region that must always exist in the direction of the galactic center, due to the tag-of-war between the Sun and the Galaxy.

My main purposes here are to point out the anomaly, describe its nature, calculate its strength, and describe some of its effects on planetary motions. I leave more detailed and systematic study to future work.

In section 2, I describe the anomaly in the frame of the modified Poisson theory. In section 3, I briefly comment on multi-potential theories, such as the nonrelativistic limit of TeVeS. In sections 4, I describe an approximation for computing the effect. In section 5, I give the numerical results for various forms of $\mu(x)$ both for numerical solutions of the full field equation and for the approximate expression. In section 6, I look at solar system effects, mainly the induced anomalous perihelion precession. Section 7 is a discussion. The appendices describe the properties of the fictitious "phantom matter" that might be viewed as giving rise to the anomaly within a Newtonian framework.

2. The quadrupole anomaly in the modified Poisson theory

The special case of the EFE studied here affects all high-acceleration galactic systems and could, in principle, be detected in precision systems such as the SS, binary pulsars, etc.. I shall concentrate here on the EFE for a point mass, because it is simpler to treat, involving, as it does, only MG, a_0, and $\eta = g_g/a_0$ as defining parameters. While the principles are similar, the treatment of a general few-body system is more complicated, because it involves several additional dimensionless parameters, such as characteristic lengths in units of the MOND radius of the system, $R_M = (MG/a_0)^{1/2}$, the mass ratios, orientation with respect to the external field, etc.. Even the

[3]In Milgrom (1986a) I discussed the anomalous potential difference that is predicted between the sun's vicinity and infinity. This, however, does not affect motions near the sun, and depends on system parameters, and on the choice of $\mu(x)$, very differently from the quadrupole anomaly discussed here. For example, for the isolated case, $\eta = 0$, the quadrupole vanishes, while the potential anomaly becomes infinite.

isolated, two-body problem is not exactly solvable in MOND, while the test particle motion in the field of a point mass is. The EFE for few-body systems will be discussed elsewhere.

Accordingly, I consider here the motion of a test particle (e.g., a planet) in the field of a point mass (e.g., the sun), embedded in an asymptotically constant gravitational field. The latter approximates, e.g., the galactic field at the position of the sun.[4] In this case, the acceleration of the test particle is given by the gradient of the modified potential field (not so for a non-test mass); so, we set to calculate the anomalous potential field for this problem.

In the modified Poisson theory of Bekenstein & Milgrom (1984), the gravitational potential, $\phi(\mathbf{r})$, due to a density $\rho(\mathbf{r})$, embedded (falling) in a constant external field $\mathbf{g}_g$, is gotten from the nonlinear generalization of the Poisson equation:

$$\vec{\nabla} \cdot [\mu(|\vec{\nabla}\phi|/a_0)\vec{\nabla}\phi] = 4\pi G\rho. \tag{2}$$

Here $\mu(x)$ is the interpolating function characterizing the theory. This equation has to be solved with the boundary conditions: $\vec{\nabla}\phi \rightarrow -\mathbf{g}_g$. The internal dynamics is then governed by the field $\vec{\nabla}\phi_{in} = \vec{\nabla}\phi + \mathbf{g}_g$. The MOND anomaly is that part of $\vec{\nabla}\phi_{in}$ in excess of the Newtonian field for the same mass when isolated, call it $-\vec{\nabla}\phi^N$. The MOND anomaly is then

$$\mathbf{g}^a = -\vec{\nabla}\phi_a = -(\vec{\nabla}\phi_{in} - \vec{\nabla}\phi^N). \tag{3}$$

We want to determine the behavior of this anomaly at distances from the Sun much smaller than R_M, where the accelerations are much higher than a_0.

The above anomalous acceleration describes all MOND effects combined. When $\delta\mu(x) = 1 - \mu(x)$ vanishes at high x as $Ax^{-\alpha}$, we get an anomaly near the sun, even in the absence of an external field. This isolated-mass acceleration is spherical and pointing towards the sun, $\mathbf{g}^a = -Aa_0(u/R_M)^{2(\alpha-1)}\mathbf{e}$, where $\mathbf{e} = \mathbf{u}/u$ is the radial unit vector and $\mathbf{u}$ is the position with respect to the Sun (Milgrom 1983). For example, for $\alpha = 1$ we get a constant anomalous acceleration pointing towards the sun $\mathbf{g}^a = -Aa_0\mathbf{e}$. This could explain the Pioneer anomaly, but produces too strong effects on the planets (as already shown by the analysis of Milgrom 1983). In a recent detailed analysis, Sereno & Jetzer (2006) conclude that SS measurements roughly allow only $\alpha \gtrsim 1.5$ (depending on the value of A).

Sereno & Jetzer take the parameter values (such as perihelion precession rate, or semi-major axis) quoted as yielding the minimum best ephemerides fit residuals, as actual limits on any possible anomalous force, not included in the fits. They thus obtain limits on the strength of such various forces, including MOND anomalies for various α. However, one may question this procedure, since the more sensible approach would be to include any suspected anomalous force in a global fit,

[4] In principle, we have to solve for the combined Galaxy-sun field. Our approximation amounts to taking the Galaxy as a mass $M \rightarrow \infty$, at a distance $R \rightarrow \infty$, with M/R^2 fixed. Corrections to this approximation are expected to be of order $(M_\odot/M)^{1/2} \sim 3 \cdot 10^{-6}$.

and in this way to determine what strength of the anomaly can be tolerated by the data (I am grateful to Gilles Esposito-Farese for making this point to me). For example, using the above criterion, Sereno and Jetzer deduce an upper limit of 10^{-10}cm s^{-2} on a constant, anomalous, sunward acceleration that can be tolerated by the data for Uranus, $4 \cdot 10^{-10}$cm s^{-2} for Neptune, and 10^{-9}cm s^{-2} for Pluto. However, Fienga et al. (2009), following the more sensible procedure, and including a constant acceleration in the equations of motion of these three planets in their global fitting procedure (to test the tolerance to a Pioneer-like anomaly), find that the Uranus data can tolerate a constant acceleration of up to $\approx 2 \cdot 10^{-8}$cm s^{-2}, and that of Neptune and Pluto allow an anomalous constant acceleration even of the size of the claimed Pioneer anomaly; namely $\approx 8 \cdot 10^{-8}$cm s^{-2}. These are some two order of magnitude larger than the limits deduced by Sereno and Jetzer (2006). Clearly, the whole question of SS constraints on the high acceleration behavior of $\mu(x)$ requires revisitation.

Be that as it may, the case $\alpha = 3/2$ happens to be interesting conceptually: It gives rise to a harmonic, anomalous acceleration $\mathbf{g}^a = -Aa_0\mathbf{u}/R_M$, which has the same u dependence, and a similar magnitude, as the quadrupole anomaly that is my main subject in this paper.

In most of what what follows I isolate the EFE from such spherical anomalies by assuming forms of μ with fast decreasing $\delta\mu(x)$ at large values of x. I will, however, discuss, in section 6, the effects of the harmonic anomaly produced in the case of the critical exponent $\alpha = 3/2$, which is interesting in the present context. It is an effect that is generic to MOND and obtain in any formulation as long as the characteristic interpolation function has the appropriate asymptotic form. The two effects are separate and largely additive, since the quadrupole anomaly depends mainly on the behavior of μ near the transition region, while the harmonic anomaly derives from the high x behavior of $\mu(x)$.

What then is $\mathbf{g}^a$ like near the Sun under the above restriction on μ? Note first, importantly, that $\mathbf{g}^a$ vanishes identically at the position of the Sun (shown in Appendix A.1). Secondly, note that with our choice of $\mu(x)$, the anomaly is practically harmonic (a solution of the Laplace equation) within a large volume around the sun. The reason for this is as follows: in this region $|\vec{\nabla}\phi|/a_0 \gg 1$ and so, practically, $\mu = 1$, there. As a result, ϕ solves the (linear) Poisson equation there, with the same density source (the sun) as ϕ^N, and hence their difference is harmonic. (For $\alpha \leq 3/2$ this is not the case: for Example, for $\alpha = 3/2$ we have a constant phantom density near the sun.) Add to these the fact that the only length scale in the problem is $R_M \approx 8 \cdot 10^3$a.u., and that the dimensionless parameter η is of order unity, and we conclude that when $|\mathbf{u}|$ is small enough with respect to R_M, $\phi_a(\mathbf{u})$ is dominated by a quadrupole (a dipole is absent because the anomalous acceleration vanishes at $\mathbf{u} = 0$); so we can write there

$$\phi_a(\mathbf{u}) \approx \phi_a(0) + \frac{1}{2}q^a_{ij}u^i u^j, \tag{4}$$

where the anomalous quadrupole is diagonal, and axisymmetric in a frame where the z axis is along the external field. Harmonicity of ϕ_a implies that q^a_{ij} is traceless:

$$q^a_{xx} = q^a_{yy} = -q^a_{zz}/2. \tag{5}$$

The effect is thus characterized by one quantity that we have to determine, say q^a_{zz}. On dimensional grounds we can write:

$$q^a_{zz} = q(\eta)a_0 R_M^{-1}, \tag{6}$$

and I seek to determine the dimensionless, quadrupole parameter $q(\eta)$, which describes the EFE anomaly.[5,6]

Near the sun, the anomaly, as defined in eq.(3), is a very small difference between the much larger $\vec{\nabla}\phi_{in}$ and $\vec{\nabla}\phi^N$. We have $|\vec{\nabla}\phi_a|/|\vec{\nabla}\phi^N| = |q|(u/R_M)[a_0/(MG/u^2)]$; the last two factors are very small (their product is $\sim 10^{-12}$ at 1a.u. from the sun), and we shall see that $q \sim 0.1$. Deducing the anomaly directly from eq.(3) by solving numerically for ϕ (e.g., using a non-linear Poison solver) could be very challenging. Now that we identified the position dependence of the anomaly, one can perhaps deduce q by evaluating the anomaly numerically at a larger distance from the Sun (e.g., at 10^3a.u. the relative strength of the anomaly is $\sim 10^{-4}$).

Even though the effect is very small in the inner solar system, the problem cannot be attacked by expanding the governing equations in a small perturbation: There is always a region where the MOND field differs greatly from the Newtonian one (see below), and the ellipticity of the modified Poisson equation ties the different regions inseparably.

There is a useful way to envisage the anomaly based on our intuition from Newtonian gravity, using the concept of "phantom mass" introduced in Milgrom (1986b). This is the density distribution, ρ_p, whose Newtonian field is the MOND anomaly. It is the mass density that a Newtonist would attribute to dark matter, and is given by

$$\rho_p \equiv \frac{1}{4\pi G}\Delta\phi_a = \frac{1}{4\pi G}\vec{\nabla}\cdot(\vec{\nabla}\phi_{in} - \vec{\nabla}\phi^N) = \frac{1}{4\pi G}\vec{\nabla}\cdot\vec{\nabla}\phi_{in} - \rho. \tag{7}$$

We can also write

$$\rho_p = \frac{1}{4\pi G}\vec{\nabla}\cdot\vec{\nabla}\phi - \rho = \frac{1}{4\pi G}\vec{\nabla}\cdot[(1-\mu)\vec{\nabla}\phi]. \tag{8}$$

The exact MOND field in the system, $-\vec{\nabla}\phi$, equals the sum of the Newtonian fields of ρ and ρ_p, and of the constant (MOND) external field.[7]

In this way, we have identified the MOND anomaly as the full field of a certain mass distribution, instead of as a very small difference between two strong fields. This can be useful for deducing the anomaly numerically, and for devising approximation schemes that will bypass the inherent difficulty of deducing a small difference of large quantities.

[5]Since the EFE anomalous acceleration is linear in u, it dominates the first type of anomaly, near the sun, when $\delta\mu \propto x^{-\alpha}$ at high x, for $\alpha > 3/2$.

[6]Note that this field differs from the field of a central quadrupole, such as an oblate sun, which has the same angular dependence but a different radial one.

[7]The MOND external field can also be viewed as the sum of the Newtonian galactic field, plus that of the galactic phantom density.

Using the field equation (2) we can write

$$\rho_p = -\frac{1}{4\pi G a_0}(\mu'/\mu)\vec{\nabla}|\vec{\nabla}\phi| \cdot \vec{\nabla}\phi + (\mu^{-1} - 1)\rho, \tag{9}$$

or in another useful form

$$\rho_p = -\frac{a_0}{4\pi G}\mathbf{e} \cdot \vec{\nabla}\mathcal{U} + (\mu^{-1} - 1)\rho, \tag{10}$$

where $\mathcal{U}(x) = \int L(x)dx$, with $L = x\mu'/\mu$ the logarithmic derivative of μ, $\mathbf{e}$ is a unit vector in the direction of $\vec{\nabla}\phi$, and $x = |\vec{\nabla}\phi|/a_0$.

We see from expression (9) that ρ_p resides where μ differs from 1, namely, in low acceleration regions. There must exist such a region, even when the external field itself is high: Near the Sun the acceleration is pointing towards the sun. As we move away in the direction of the galactic center, the acceleration must eventually vanish at some point, and change sign to point in the opposite direction. There is thus always a region $\mathcal{P}$ around the critical point that is in the sub a_0 regime, in which MOND acts in full force, and in which ρ_p is not negligible. This region excludes a large volume around the Sun with our choice of μ. Even with the limiting form of μ, where $\mu = 1$ for $x > 1$, the region $\mathcal{P}$, the distribution of ρ_p in it, and thus the anomaly that is felt near the sun, remain finite.

The anomaly can be written as

$$\mathbf{g}^a(\mathbf{R}) = G\int \frac{\rho_p(\mathbf{r})(\mathbf{r} - \mathbf{R})}{|\mathbf{r} - \mathbf{R}|^3} d^3\mathbf{r}. \tag{11}$$

We thus identify[8]

$$q^a_{zz} = -\left[\frac{dg^a_z}{dZ}\right]_{\odot} = G\int \frac{d^3u}{u^3}\rho_p(\mathbf{u})(1 - 3\cos^2\theta), \tag{12}$$

where $\mathbf{u} = \mathbf{r} - \mathbf{R}_0$ is the position from the sun, and $\cos\theta = n_z \equiv u_z/u$.

In Appendix A, I derive several useful, exact properties of the phantom mass. First, I show that around the critical point there are always regions of both positive and negative phantom density (A.3) (this is an example of the general trend discussed in Milgrom 1986b). Second, in A.4 I derive the total phantom mass M_p; it depends on η and the form of μ. (For the limiting form of μ we have $M_p = 0$ identically; so the negative and positive densities exactly cancel) . Third, I derive the column density along the symmetry axis (A.2). It is independent of η, and for the limiting form of μ it is $a_0/2\pi G$. Also, I find [in A.6] that ρ_p decreases asymptotically as u^{-3} (taking both signs at all radii); so this provides good convergence in eq.(12).

[8]This can be used only when ρ_p vanishes at the origin, as I assume all along. Otherwise the u integration diverges at the origin and the integral is undefined. The quantity this integral describes can still be finite when ρ_p doesn't vanish at the origin.

I also derive in the appendix [eq.(A12)] an expression for the anomaly as an integral over $\vec{\nabla}\phi$ (not involving derivatives of $\vec{\nabla}\phi$ as appear in ρ_p):

$$\mathbf{g}^a(\mathbf{R}) = \frac{1}{4\pi}\int \frac{(\mu-1)}{|\mathbf{r}-\mathbf{R}|^3}[\vec{\nabla}\phi - 3(\vec{\nabla}\phi\cdot\mathbf{n})\mathbf{n}]d^3\mathbf{r} - \frac{1}{3}(1-\mu_g)\mathbf{g}_g, \tag{13}$$

where $\mu_g = \mu(\eta)$ is the galactic value of μ, and $\mathbf{n}$ is a unit vector in the direction of $\mathbf{r}-\mathbf{R}$. Although eq.(13) is equivalent to eq.(3), it is a better starting point for approximations. The great sensitivity of the anomalous acceleration, as deduced from eq.(3), to errors in $\vec{\nabla}\phi$ (or equivalently in $\vec{\nabla}\phi_{in}$), still remain in expression (13), but here it is controllable. Errors in the integration, will cause $\mathbf{g}^a(\mathbf{R})$ to vanish not at the position of the sun, but at a somewhat different point off by $\delta\mathbf{R}$. Even if $|\delta\mathbf{R}|/R_M \ll 1$, the resulting errors in $\mathbf{g}^a(\mathbf{R})$ near the Sun could be very large. However, we can eliminate this problem by considering directly[9]

$$q^a_{zz} = -\left[\frac{dg^a_z}{dZ}\right]_\odot = -\frac{1}{4\pi}\int \frac{(\mu-1)}{|\mathbf{u}|^4}[6\phi_{,z}n_z + 3\vec{\nabla}\phi\cdot\mathbf{n}(1-5n_z^2)]d^3u, \tag{14}$$

where $\mathbf{n}$ is now a unit vector in the direction of $\mathbf{u}$. If in eq.(14) we take $\mathbf{u}$ in units of R_M and $\vec{\nabla}\phi$ in units of a_0, the expression yields q.

The phantom mass is only a visualization aid that I find very convenient. We could have derived eqs.(13) and (14), simply by starting from the identity (for functions that decay fast enough at infinity):

$$\phi_a(\mathbf{R}) = -\frac{1}{4\pi}\int \frac{\Delta\phi_a(\mathbf{r})}{|\mathbf{r}-\mathbf{R}|}d^3\mathbf{r} = -\frac{1}{4\pi}\int \frac{\vec{\nabla}\cdot[(1-\mu)\vec{\nabla}\phi](\mathbf{r})}{|\mathbf{r}-\mathbf{R}|}d^3\mathbf{r}, \tag{15}$$

(where the second equality comes from the definition of ϕ_a), and proceed from there, integrating by parts, without mentioning phantom matter. The concept is, however, also useful because my main approximation is to replace the true phantom matter by a surrogate one, which I do in section 4.

To recapitulate, eqs.(11)-(14) are only identities, useful here in two ways: a. for calculating the quadrupole q^a_{zz} as an integral over "large" quantities, instead of as a very small difference between much larger quantities, b. for devising approximations for q^a_{zz}.

To use eq.(14) to calculate the anomaly we need to solve the nonlinear Poisson equation for the required choice of interpolating function and an η value, and use the resulting field in eq.(14) (instead of subtracting the Newtonian acceleration from the numerical solution very near the sun).

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[9]This expression converges well near the Sun despite the $|\mathbf{u}|^4$ factor, and the divergence of $\vec{\nabla}\phi$, because $\mu-1$ vanishes quickly there (and because the field becomes spherical, for which case the angular integral vanishes). When $\delta\mu(x)$ vanishes faster than $x^{-3/2}$ for $x\to\infty$, the u integration is convergent near $u=0$, and we can use this expression without further care. For $\delta\mu(x)\to x^{-3/2}$, the u integral diverges logarithmically; however, the angular integral vanishes, and the z derivative of the force is still finite at the sun. In this case, the anomalous field near the sun is not the pure quadrupole, as I mentioned above. At infinity, the convergence is also good, because there $\vec{\nabla}\phi \to -\mathbf{g}_g + \vec{\nabla}\delta$ with $\vec{\nabla}\delta$ decreasing like u^{-2}. For the constant part the angular integral vanishes, so there is convergence at least as d^3u/u^6. See also eq.(12)

It is useful, however, to develop an approximation scheme that will permit a quick calculation of q for mass-production studies, since μ and the relevant η for the solar system are not exactly known, and also for application to other precision systems, perhaps involving more massive bodies. I shall then resort to an approximation that will replace ρ_p with a reasonable surrogate, $\tilde{\rho}_p$, that is known in closed form. The surrogate anomaly can then be written as an integral of a known function.

3. Multi-potential theories

It is interesting to study the EFE in the SS in other modified-gravity formulations of MOND, such as multi-potential, modified-Poisson theories. In these, the gravitational potential is the sum of several potentials, each satisfying a modified Poisson equation of its own. For example, the nonrelativistic limit of TeVeS is a double-potential theory: the potential is the sum of a Poissonian potential and one that is a solution of the nonlinear Poisson equation, both having the true density as source. Generalizing this, consider a nonrelativistic theory in which a particle's equation of motion is $\mathbf{a} = -\vec{\nabla}\Phi$, with $\Phi = \sum_i \phi_i$, and with ϕ_i determined from

$$\vec{\nabla} \cdot [\mu_i(|\vec{\nabla}\phi_i|/\alpha_i a_0)\vec{\nabla}\phi_i] = 4\pi\beta_i G\rho. \tag{16}$$

For concreteness sake, assume that in the limit $x \to \infty$ all $\mu_i(x)$ go to constants[10], and take β_i such that all these constants are 1. (See relevant comments in Zhao & Famaey 2006, Famaey & al. 2007, Bruneton & esposito-Farese 2007, and Milgrom & Sanders 2008, who deal with constraints on the form of the interpolating function in TeVeS.) Also take G to be the usual gravitational constant, which means that $\sum_i \beta_i = 1$. The dimensionless coefficients α_i can be absorbed into the definition of μ_i, but it is convenient to keep them so that we can require a uniform low-x behavior $\mu_i(x) \to x$. In this specific case, brought as an example, we have in the very-low-acceleration limit $|\vec{\nabla}\phi_i| \ll \alpha_i a_0$, for a spherical system, $|\vec{\nabla}\phi_i| \approx (\alpha_i\beta_i)^{1/2}(a_0|\vec{\nabla}\phi^N|)^{1/2}$, and a_0 then plays its usual role in phenomenology if we normalize it so that $\sum_i(\alpha_i\beta_i)^{1/2} = 1$ (I assume that $\alpha_i, \beta_i \geq 0$, so all potentials describe attractive gravity). If one of the $\alpha_i \to 0$, the corresponding $\mu_i(x) \to 1$ for all x; so, ϕ_i satisfies the linear Poisson equation $\vec{\nabla} \cdot \vec{\nabla}\phi_i = 4\pi\beta_i G\rho$. For example, the nonrelativistic limit of TeVeS would correspond to $\alpha_1 \to 0$, and $\mu_2 = \mu$ and $\beta_2 = \beta$ still unspecified ($\beta_1 = 1 - \beta$, $\alpha_2 = \beta^{-1}$).

Each of the sub-potentials ϕ_i is solved for separately, each generates its own cloud of "phantom matter" (which do not interact with other clouds), each creating its own sub-anomaly

$$\mathbf{g}_i^a = -\vec{\nabla}(\phi_i - \beta_i\phi^N) \tag{17}$$

(ϕ^N is the standard Newtonian potential, calculated with the standard G).

To apply our results to the multi-potential case, we first have to decompose the galactic field at the system's (e.g., the sun's) position to its different components (for example, by solving the

[10] In principle we can have more general behaviors, but I do not wish to delve into a general discussion here.

full problem for the Galaxy's mass distribution, determining the values of the sub-fields at that position): $\mathbf{g}_g = \sum_i \mathbf{g}_g^i$. Then, remembering that the effective constants for each potential are $G_i = \beta_i G$, $a_0^i = \alpha_i a_0$, and $R_M^i = (\beta_i/\alpha_i)^{1/2} R_M$, we define $\eta_i = |\mathbf{g}_g^i|/\alpha_i a_0$. We then calculate the dimensionless quadrupole anomaly $q_i(\eta_i)$ using our results above and below. We can also, as an approximation that is exact for a spherical galaxy, use the algebraic relations

$$\mathbf{g}_g^i = \beta_i \nu(\beta_i |\mathbf{g}_N^g|/\alpha_i a_0)\mathbf{g}_N^g, \tag{18}$$

to determine $\mathbf{g}_g^i$ from the Newtonian galactic field $\mathbf{g}_N^g$.

The Sun is at a position in the Galaxy with a symmetric perspective. Under these simplifying circumstances, all the $\mathbf{g}_g^i$ must point approximately in the same direction: towards the galactic center. The anomaly's symmetry axis is then common to all the sub anomalies, and we simply have, again, a biaxial, quadrupole anomaly with $-2q_{xx}^a = -2q_{yy}^a = q_{zz}^a = qa_0/R_M$, and

$$q = \sum_i q_i(\eta_i)\alpha_i^{3/2}\beta_i^{-1/2}. \tag{19}$$

For a planetary system placed at an arbitrary position in the Galaxy, the directions of $\mathbf{g}_g^i$ may be different. We then straightforwardly add up the different quadrupole sub anomalies. The result is still a traceless quadrupole; but it is, in general, triaxial, with all three principal axes, and elements, determined by the $q_i(\eta_i)$ and the respective directions of $\mathbf{g}_g^i$.

For TeVeS, the Poissonian field, ϕ_1, does not create an anomaly. The galactic field $\mathbf{g}_g^2$ defines the symmetry axis, even for an arbitrary location in the Galaxy. The anomaly is thus always a biaxial quadrupole, with $q = \beta^{-2} q_2(\eta_2)$.

4. The surrogate phantom density and its properties

As an approximation, I shall substitute for ρ_p a surrogate phantom density, $\tilde{\rho}_p$, derived from the divergence of the MOND field, as approximated by the algebraic relation eq.(1). Call this field $\mathbf{g}^*$, then

$$\tilde{\rho}_p = -\frac{1}{4\pi G}\vec{\nabla}\cdot\mathbf{g}^* - \rho = -\frac{1}{4\pi G}\vec{\nabla}\cdot(\mathbf{g}^* - \mathbf{g}_N), \tag{20}$$

with $\mathbf{g}^* = \nu(|\mathbf{g}_N|/a_0)\mathbf{g}_N$, and

$$\mathbf{g}_N = -\eta_N a_0 \mathbf{e}_z - \frac{M_\odot G}{|\mathbf{u}|^3}\mathbf{u} = -a_0(\eta_N \mathbf{e}_z + \frac{\hat{\mathbf{u}}}{|\hat{\mathbf{u}}|^3}), \tag{21}$$

where $\hat{\mathbf{u}} = \mathbf{u}/R_M$ (the galactic field is in the $-z$ direction), and η_N is the Newtonian value of η. We thus have ρ_p in closed form.[11]

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[11]Note that I do not use $\mathbf{g}^*$ as the MOND field; i.e., I do not substitute it for $\vec{\nabla}\phi$ for use in eq.(3). I only use it to obtain an approximation for ρ_p. The Newtonian field of $\tilde{\rho}_p$; i.e., the surrogate anomaly $\tilde{\mathbf{g}}^a$, which is derivable

All the results based on the surrogate density depend on the Newtonian galactic field at the sun's position, $\mathbf{g}_g^N$, and do not require knowledge of the MOND value of this field, $\mathbf{g}_g$, while the results using the exact theory depend on the latter alone. The relation between these two accelerations depends on the galactic mass distribution, which we don't know exactly (we can measure $\mathbf{g}_g$, but need a mass model to determine $\mathbf{g}_g^N$). We do know that because the Sun is very near the galactic mid-plane, symmetry dictates that the two are very nearly parallel. I assume in all that follows that they are, and that their values are related by $\eta_N = \mu(\eta)\eta$, which is exact for a spherical galaxy. This is a good approximation near the sun, and, in any event, has little consequence for us here, since it is only used to relate the value of η_N to be used in the surrogate-density approximation to the value of η appropriate for the sun: the small uncertainty in $\eta_N(\eta)$ is rater immaterial.

The surrogate phantom density $\tilde{\rho}_p$ is somewhat different from ρ_p. However, $\tilde{\rho}_p$ does have properties (demonstrated in Appendix B) that are rather similar to those of ρ_p, on which we can found the hope that this is a reasonable approximation for calculating the anomaly: First, $\tilde{\rho}_p$ has a total mass similar to that of ρ_p (for the limiting form of μ they both vanish) (B.4). Second, $\tilde{\rho}_p$ has a similar pattern of sign distribution (B.3). Third, the column density of $\tilde{\rho}_p$ along the symmetry axis is identical to that of ρ_p (B.2). Fourth, the Newtonian field of $\tilde{\rho}_p$, like that of ρ_p, vanishes at the position of the sun, and is thus also a traceless, quadrupole near the Sun (B.1).

In some regards, the surrogate density does not quite mimic ρ_p. For example, as can be seen from appendices A.6, and B.6, the angular distributions of their asymptotic behavior (for $u \gg \eta^{-1/2}R_M$) are different.

As shown in A.5, substituting $\tilde{\rho}_p$ for ρ_p is tantamount to replacing $\vec{\nabla}\phi$ in eq.(13) by $-\mathbf{g}^*$. This gives for the surrogate anomaly:

$$\tilde{\mathbf{g}}^a(\mathbf{R}) = \frac{1}{4\pi}\int \frac{(\nu-1)}{|\mathbf{r}-\mathbf{R}|^3}[\mathbf{g}_N - 3(\mathbf{g}_N\cdot\mathbf{n})\mathbf{n}]d^3\mathbf{r} - \frac{1}{3}(\nu_g-1)\mathbf{g}_g^N. \tag{22}$$

Here $\nu_g = \nu(\eta_N)$ is the galactic value of ν $[(\nu_g-1)\mathbf{g}_g^N = (1-\mu_g)\mathbf{g}_g]$. From this follows the expression for the dimensionless, surrogate, anomalous quadrupole:

$$\tilde{q}(\eta) = -\frac{1}{4\pi}\int \frac{(\nu-1)}{|\mathbf{u}|^4}[6g_z^N n_z + 3\mathbf{g}_N\cdot\mathbf{n}(1-5n_z^2)]d^3u, \tag{23}$$

with $\mathbf{g}_N$ from eq.(21), now expressed in units of a_0, and $\mathbf{u}$ in units of R_M. We can write in spherical

from a potential, is not equal to $\mathbf{g}^* - \mathbf{g}_N$, even though the two fields have the same divergence everywhere; $\mathbf{g}^*$ is not, generally, derivable from a potential. The two differ by a curl field.

coordinates[12]

$$\tilde{q}(\eta)=\frac{3}{2}\int_0^\infty\frac{du}{u^2}\int_{-1}^1 d\xi(\nu-1)[\eta_N(3\xi-5\xi^3)+u^{-2}(1-3\xi^2)]=$$

$$=-3\int_0^\infty\frac{du}{u^2}\int_{-1}^1 d\xi(\nu-1)[\eta_N P_3(\xi)+u^{-2}P_2(\xi)], \quad (24)$$

where P_ℓ is the ℓs Legendre polynomial, and with $\nu=\nu[(\eta_N^2+u^{-4}+2\eta_N u^{-2}\xi)^{1/2}]$ ($\xi=\cos\theta$). Because the angular integral vanishes when the factor $\nu-1$ in the integrand is ξ independent, we can replace the 1 in $\nu-1$ with a function of u alone, without changing the result. For example, we can replace this 1 with the value of ν for $\xi=0$, i.e., with $\nu[(\eta_N^2+u^{-4})^{1/2}]$, or by the value of ν at $\xi=\pm1$: $\nu(|\eta_N\pm u^{-2}|)$. With these, convergence is improved also at large u, where $\nu\rightarrow\nu(\eta_N)\neq1$. Changing variables to $v=u^{-1}$ we can also write

$$\tilde{q}(\eta)=-3\int_0^\infty dv\int_{-1}^1 d\xi(\nu-1)[\eta_N P_3(\xi)+v^2P_2(\xi)], \quad (25)$$

with $\nu=\nu[(\eta_N^2+v^4+2\eta_N v^2\xi)^{1/2}]$.

In Appendix C, I consider further one particular case: the limiting form $\mu(x)=x$ for $x\leq1$, and $\mu(x)=1$ for $x>1$. This is useful pedagogically, since it lands itself to further approximation that will lead to a closed expression for the surrogate anomaly. This can be used, for example, as a touchstone for checking numerical results. It also affords a simple description of the phantom matter; and, it also seems to give a lower limit to the anomaly for a large class of interpolating functions.

For this limiting form of μ, with $\eta_N>1$ (which is relevant for the SS), we have a well demarcated region $\tilde{\mathcal{P}}$ outside which $\tilde{\rho}_p=0$: We have at large distances in all directions $|\vec{\nabla}\phi|\rightarrow|\mathbf{g}_g|>a_0$, which takes us outside $\tilde{\mathcal{P}}$; so $\tilde{\mathcal{P}}$ is bounded in all directions. The boundaries of $\tilde{\mathcal{P}}$ can then be obtained in closed form, and are described in B.7. For $\eta_N<1$, or for choices of $\mu(x)$ that reach the value 1 only asymptotically at high x, the region $\tilde{\mathcal{P}}$ is not bounded.

The further approximation that I apply to the limiting case can be justified in the limit of large $\eta=\eta_N$, and leads to [see eq.(C8)]

$$\hat{q}(\eta)=-\eta^{-7/2}/30. \quad (26)$$

[12]This expression can be used when $\delta\mu(x)$ vanishes faster than $x^{-3/2}$ for $x\rightarrow\infty$ (equivalent to $\nu-1$ vanishing faster than $y^{-3/2}$). Otherwise, the u integral diverges and we are not justified in choosing this order of integration. For example, for the isolated, spherical case ($\eta_N=0$), this expression vanishes for any choice of μ, which is a wrong result when $\delta\mu(x)$ vanishes as $x^{-3/2}$, or slower.

5. Results for the dimensionless quadrupole anomaly

I use eq.(24), or eq.(25), to compute the dimensionless, surrogate quadrupole anomaly $\tilde{q}(\eta)$ for various forms of $\mu(x)$ and different η values.[13] I also wrote a simple nonlinear-Poisson solver (based on the detailed blueprints described in Milgrom 1986a) to calculate q for several control cases from the exact expression (14).

I consider three one-parameter families of interpolating functions, defined and discussed in Milgrom & Sanders (2008), and checked to give good results in rotation curve analysis in the transition region. They also have a rapid decline of $\delta\mu(x)$ at high x, with a negligible anomaly resulting from the finiteness of $\delta\mu(x)$. The first family is

$$\mu_\alpha(x) = \frac{x}{(1+x^\alpha)^{1/\alpha}}, \tag{27}$$

with the corresponding

$$\nu_\alpha(y) = \left[\frac{1+(1+4y^{-\alpha})^{1/2}}{2}\right]^{1/\alpha}, \tag{28}$$

which I use with $\alpha > 3/2$. The limiting form of $\mu(x)$ corresponds to very large α.

The second and third families are defined by their $\nu(y)$ functions:

$$\tilde{\nu}_\alpha(y) = (1-e^{-y})^{-1/2} + \alpha e^{-y}, \tag{29}$$

and

$$\bar{\nu}_\alpha(y) \equiv (1-e^{-y^\alpha})^{(-1/2\alpha)} + (1-1/2\alpha)e^{-y^\alpha}, \tag{30}$$

with the corresponding $\tilde{\mu}_\alpha(x)$, and $\bar{\mu}_\alpha$. The choice of α permits a very fast transition to the Newtonian regime near $y = 1$. These choices are only indicative; they are neither fully representative, nor exhaustive.

Table 1 gives the numerical results for $-\tilde{q}$. For some entries I also give in parentheses the value obtained from the numerical solution of the modified Poisson equation via eq.(14).

The value of η for the galactic field in the solar neighborhood is given by

$$\eta \approx 1.9\left(\frac{V_\odot}{220\text{km s}^{-1}}\right)^2\left(\frac{R_\odot}{8\text{Kpc}}\right)^{-1}\left(\frac{a_0}{10^{-8}\text{cm s}^{-2}}\right)^{-1}, \tag{31}$$

[13] As a check of the numerics I also use eq.(22) to calculate the z component of the anomalous acceleration along the z axis, near the sun, to ascertain that is vanishes at the sun's position, and to check how good the quadrupole approximation is away from the sun. Both aspects check well; for example, the departure from a pure quadrupole behavior is, as expected, of the order or u/R_M, only about one percent within 80a.u..

with $V_\odot$ the orbital velocity of the Sun in the Galaxy, and $R_\odot$ its orbital radius.[14,15] The relevant range for our problem is thus roughly $1 \lesssim \eta \lesssim 2$. In the table this gives a range $0.01 \lesssim -\tilde{q} \lesssim 0.3$. The interpolating functions in the last two rows are rather less likely; if we omit those the range shrinks to $0.07 \lesssim -\tilde{q} \lesssim 0.3$.

We see that the surrogate approximation works rather well, and by and large produces $\tilde{q}$ values that do not differ from q by much. We also see that that expression (26) is a rather good approximation, in the limiting case, down to η values of 1.5, even though it can be justified as an approximation only for large values of η.

Notice that $|\tilde{q}|$ decreases for increasing η, when $\eta \gg 1$, because the phantom matter becomes less and less important; $|\tilde{q}|$ also decreases with decreasing η, for $\eta < 1$, because the phantom matter becomes more and more spherical. The maximum effect seems, in fact, to occur for the range of η values around that for the galactic field near the sun.

6. The effect of the anomalous quadrupole on planetary motions

For $|\mathbf{u}| \ll R_M$, the quadrupole anomaly due to the MOND EFE is of the same form as that produced by tidal effects of a static distant mass[16] (such as a molecular clouds, a nearby star, etc.). For example, a solar mass star at a distance d is expected to give $q^*_{zz} \approx -4 \cdot 10^{-5}(d/\mathrm{pc})^{-3} a_0/R_M \approx -4 \cdot 10^{-4}(q/0.1)^{-1}(d/\mathrm{pc})^{-3} q^a_{zz}$. The tidal effect of the Galaxy is dominated by the gradient in the direction perpendicular to the disc, call it the x axis, and we have for its quadrupole component $q^{tide}_{xx} \approx 4\pi\rho_0$, where $\rho_0 \approx 0.076 M_\odot/\mathrm{pc}^3$ (Crézé & al. 1997) is the midplane dynamical density of the Galaxy at the sun's position. We thus have $q^{tide}_{xx} \approx 3 \cdot 10^{-4}(q/0.1)^{-1} q^a_{zz}$. The other two quadrupole components of the galactic tidal field are about an order of magnitude smaller. Such effects are thus much smaller than the EFE quadrupole anomaly.

One may wonder what the MOND effects are of the planets on each other. Beside the standard physics, mutual effects, which are always taken into account, there are new MOND effects here as well. Again, some of these have to do with the remaining small value of $1-\mu(x)$ at high x, which we ignore here. The effects that remain even with the limiting form of μ can, again, be described as the Newtonian effects of the phantom matter. For any given configuration of the SS there are several

[14] The value of a_0 is determined from several of its appearances in the phenomenology to be around $10^{-8}\mathrm{cm\ s}^{-2}$. For example, Begeman, Broeils, & Sanders (1991) found from rotation curve analysis $a_0 \approx 1.2 \cdot 10^{-8}\mathrm{cm\ s}^{-2}$. Recently, Stark, McGaugh, & Swaters (2009) found that gas dominated galaxies satisfy the mass-asymptotic-rotational-velocity relation predicted by MOND–aka the baryonic Tully-Fisher relation–($M = a_0^{-1} G^{-1} V_\infty^4$) with $a_0 = 1.2 \cdot 10^{-8}\mathrm{cm\ s}^{-2}$, and a range of $(0.7-2) \cdot 10^{-8}\mathrm{cm\ s}^{-2}$.

[15] When dealing with multi-field theories rather different η values may appear.

[16] At radii of order R_M or larger the effect is very different from tidal effects, being the Newtonian field of ρ_p. So, for example, it decreases beyond $\sim R_M$.

critical points, each begetting its own cloud of phantom matter; for example, one near each planet due to its competition with the Sun (Bekenstein & Magueijo 2006 consider the potential for probing such regions directly). There is also some distortion of the main phantom matter distribution near the solar-galactic critical point. I intend to discuss the problem of a few-body system elsewhere. Here I only mention that these effects are negligible in the present case.

To test for the presence of the quadrupole anomaly, and constrain its parameters, one needs to incorporate the anomalous force in the equations of motion in programs that fit for the observed motions of the planets. Today, the direction to the galactic center, which we take to define our symmetry axis, is only about 6 degrees away from the ecliptic. To first approximation, it can thus be assumed that our symmetry axis is in the planetary orbital plane, and take the x axis to also lie in this plane.[17] Then, the anomalous acceleration in the orbital, $x-z$ plane, is

$$\mathbf{g}^a = \frac{q^a_{zz}}{2}(x, -2z), \tag{32}$$

at position $\mathbf{u} = (x, z)$ from the sun.

To get an idea of the size of the expected effects of the anomaly on the planetary motions, consider, for example, the anomalous perihelion advance rate it produces. Let

$$u = \frac{a(1-e^2)}{1 + e\ \cos\psi}, \tag{33}$$

be the unperturbed Keplerian orbit of a planet, with the azimuthal angle ψ (which increases with time) measured from perihelion; (a and e are the semi-major axis and eccentricity, respectively).

One can use the Gauss equations to calculate the variation of the orbital elements under the action of a given acceleration field (Brouwer & Clemence 1961). For this, one needs the instantaneous radial, tangential, and normal components of the acceleration, g^a_r, g^a_t, (in the direction of increasing ψ), and g^a_n, respectively. In our case they are

$$g^a_r = \frac{q a_0 u}{2R_M}[1 - 3\cos^2(\psi - \psi_0)], \tag{34}$$

$$g^a_t = \frac{3q a_0 u}{2R_M}\sin(\psi - \psi_0)\cos(\psi - \psi_0), \tag{35}$$

and I take g^a_n to vanish (in the ecliptic it is about an order smaller than the other components). Here ψ_0 is the value of ψ on the positive z axis. With $g^a_n = 0$, there is no precession of the orbital plane, and the instantaneous perihelion advance, for example, can be calculated as

$$\dot{\omega} = \frac{(1-e^2)^{1/2}}{nea}[-g^a_r\cos\psi + g^a_t(1 + \frac{u}{p})\sin\psi], \tag{36}$$

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[17]This is not a good assumption for Pluto and Icarus, which I also consider. It also breaks down at other phases in the galactic rotation.

where ω is the longitude of perihelion of the perturbed trajectory, $n = (M_\odot G/a^3)^{1/2}$, and $p = a(1-e^2)$. To get the mean perihelion advance rate we average this rate over a period. I find

$$\langle\dot{\omega}\rangle = -\frac{qa_0}{nR_M}\kappa(e)\lambda(\psi_0), \tag{37}$$

where

$$\kappa(e) = (1-e^2)^3\left(\frac{1+s^2}{1-s^2}\right)^5, \qquad \lambda(\psi_0) = \frac{3}{8}[1+5\cos(2\psi_0)], \tag{38}$$

with $s \equiv [1-(1-e^2)^{1/2}]/e$.

So,

$$\langle\dot{\omega}\rangle \approx 4.4\cdot 10^{-20}\kappa(e)\lambda(\psi_0)\left(\frac{-q}{0.1}\right)\left(\frac{a_0}{10^{-8}\text{cm s}^{-2}}\right)^{3/2}\left(\frac{a}{\text{a.u.}}\right)^{3/2}\text{s}^{-1} =$$
$$= 2.8\cdot 10^{-5}\kappa(e)\lambda(\psi_0)\left(\frac{-q}{0.1}\right)\left(\frac{a_0}{10^{-8}\text{cm s}^{-2}}\right)^{3/2}\left(\frac{a}{\text{a.u.}}\right)^{3/2}\text{"/c}. \tag{39}$$

Note that, depending on ψ_0, the effect can take up both signs, as $\lambda(\psi_0)$ can vary between $-3/2$ and $9/4$.

I also calculated the predicted rate of secular changes in the semi-major axis due to the quadrupole anomaly and found that it vanishes for all values of e and ψ_0.

The $\kappa\lambda$ values, and the predicted anomalous perihelion precession rates for the planets, and for Icarus, are shown in Table 2.

Also shown, under $|\Delta\dot{\omega}|$, are some indicative intervals for $\dot{\omega}$ from published best-fit results for the precession rates, obtained through fitting to specific conventional models. They are shown as an indication of the level of accuracy that has been reached for the different planets. The tightest values are for Earth and Mars and, for our reference value of $q = 0.1$, are $\sim 8-10$ times larger than the expected rates for the two planets. So, parameter values that give $q \sim 0.3$, and $a_0 = 1.2\cdot 10^{-8}\text{cm s}^{-2}$, could already be only a factor 2-3 below these intervals.[18]

For the outer planets, the effect is expected to be larger, but so are the measurement errors, at present. So the prospects of first constraining the anomalous MOND EFE would seem to come if the measurements for the outer planets can be improved.

It is interesting to compare the above results with what we get from the harmonic anomaly predicted in the inner SS–even without an external field–if the asymptotic behavior of the interpolating function is $\mu(x) \to 1 - Ax^{-3/2}$. In this case we get generically in MOND, *an additional*

[18] Fienga & al. (2009) quote Pitieva as giving a value of $(-6\pm 2)\cdot 10^{-3}$"/c, for the residual perihelion precession of Saturn, which might be viewed, formally, as a positive detections of an anomaly. Other analyses they quote give $(-10\pm 8)\cdot 10^{-3}$"/c. Such an anomaly could be explained, e.g., by the harmonic anomaly I discuss below. However, since these authors seem not to view these results as definitely indicating a detection of an anomaly, I took it as an indicative accuracy of $|\Delta\dot{\omega}| \sim 10^{-2}$"/c for Saturn.

anomaly near the sun, with

$$g_r^h = \frac{-Aa_0u}{R_M}, \tag{40}$$

and with $g_t^h = 0$. It is thus of the same order, and has the same distance dependence as the quadrupole anomaly. Using these in the Gauss equations we get a secular perihelion rate:

$$\langle\dot{\omega}^h\rangle = -\frac{3Aa_0}{2nR_M}\kappa(e), \tag{41}$$

or,

$$\langle\dot{\omega}^h\rangle \approx -6.5\cdot 10^{-19}A\kappa(e)\left(\frac{a_0}{10^{-8}\text{cm s}^{-2}}\right)^{3/2}\left(\frac{a}{\text{a.u.}}\right)^{3/2}\text{s}^{-1} =$$
$$= -4.1\cdot 10^{-4}A\kappa(e)\left(\frac{a_0}{10^{-8}\text{cm s}^{-2}}\right)^{3/2}\left(\frac{a}{\text{a.u.}}\right)^{3/2}\text{"/c}. \tag{42}$$

I give in Table 2 the predicted rates for this effect for the value $A = 2/3$, which corresponds, for example, to the form $\mu_{3/2} = x(1+x^{3/2})^{-2/3}$, which has the appropriate asymptotic form.

7. Discussion

In the framework of the modified-Poisson formulation of MOND, one expects an anomalous quadrupole acceleration $g_i^a(\mathbf{u}) = -q_{ij}^a u^j$, at position $\mathbf{u}$ with respect to the sun, with q_{ij}^a diagonal, axisymmetric, and traceless: $-2q_{xx}^a = -2q_{yy}^a = q_{zz}^a = q(\eta)(a_0/R_M)$. Interestingly, this anomaly remains finite even if the MOND interpolating function is arbitrarily close to unity at high accelerations, despite the fact that the accelerations near the sun are $\gg a_0$. I find numerically $|q| \sim 0.1$, and this corresponds to an anomalous acceleration $\sim 10^{-5}a_0$ for the inner planets, but $\sim 10^{-4}a_0$ for Saturn, and even larger for the yet farther planets. These do not conflict with published SS bounds on anomalous perihelion precession, but useful constraints seem to be within reach.

This effect can be conveniently envisaged as the tidal effect on the solar system, of the imaginary phantom mass that, according to MOND, hovers around the sun.

In all the examples I studied, q turned out to be negative (repulsion from the Sun along the z axis and attraction in the perpendicular direction). I was not able to prove that this is always the case from basic properties of μ. This question of the sign of q requires a more systematic, extensive study.

Our results here also apply to the unusual sort of gravitationally polarizable matter proposed by Blanchet & Le Tiec (2008, 2009) as a model for MOND, as it reproduces the potential of the nonlinear Poisson equation.

I find that the strength of the effect is sensitive to the form of the interpolating function $\mu(x)$ in the MOND-Newtonian transition region ($x \sim 1$). The constraints we have on this aspect of μ comes mainly from rotation-curve analysis. Different studies–e.g. by Famaey & Binney (2005), Zhao &

$\eta \rightarrow$	0.5	1	1.5	2	3	5	10
$\bar{\mu}_2$	$3.4 \cdot 10^{-2}$	0.12	0.20	$9.8 \cdot 10^{-2}$	$2.5 \cdot 10^{-2}$	$3.8 \cdot 10^{-3}$	$3.1 \cdot 10^{-4}$
$\bar{\mu}_{1.5}$	$3.5 \cdot 10^{-2}$	0.11	0.19	0.16	$6.3 \cdot 10^{-2}$	$1.1 \cdot 10^{-2}$	$9.2 \cdot 10^{-4}$
$\tilde{\mu}_{.5}$	$3.6 \cdot 10^{-2}$	0.11	0.17	0.21	0.18	$9.0 \cdot 10^{-2}$	$1.2 \cdot 10^{-2}$
		(0.16)	(0.21)	(0.24)	(0.20)		
$\tilde{\mu}_1$	$3.2 \cdot 10^{-2}$	0.11	0.21	0.28	0.28	0.13	$1.8 \cdot 10^{-2}$
μ_2	$3.9 \cdot 10^{-2}$	$8.8 \cdot 10^{-2}$	0.11	0.12	0.12	0.11	$8.2 \cdot 10^{-2}$
		(0.10)	(0.12)	(0.12)	(0.12)		
μ_3	$4.0 \cdot 10^{-2}$	$8.1 \cdot 10^{-2}$	$7.9 \cdot 10^{-2}$	$6.5 \cdot 10^{-2}$	$4.4 \cdot 10^{-2}$	$2.3 \cdot 10^{-2}$	$8.8 \cdot 10^{-3}$
μ_∞	$4.3 \cdot 10^{-2}$	$4.4 \cdot 10^{-2}$	$8.8 \cdot 10^{-3}$	$3.1 \cdot 10^{-3}$	$7.3 \cdot 10^{-4}$	$1.2 \cdot 10^{-4}$	$1.1 \cdot 10^{-5}$
		$(5.3 \cdot 10^{-2})$	$(1.1 \cdot 10^{-2})$	$(4.6 \cdot 10^{-3})$			
eq.(26)	4.3	$3.3 \cdot 10^{-2}$	$8.1 \cdot 10^{-3}$	$2.9 \cdot 10^{-3}$	$7.1 \cdot 10^{-4}$	$1.2 \cdot 10^{-4}$	$1.1 \cdot 10^{-5}$

Table 1: Values of the dimensionless, surrogate, quadrupole anomaly $-\tilde{q}$ for various choices of the interpolating function and various values of η. In parentheses are shown some values of $-q$ calculated using numerical solutions of the exact modified Poisson equation.

	Merc.	Ven.	Ea.	Mar.	Jup.	Sat.	Uran.	Nept.	Plut.	Icar.
$\kappa\lambda$	2.1	.38	1.98	-1.0	-1.1	2.2	-1.5	.56	.50	$-.18$
$\langle\dot{\omega}\rangle$	.13	.065	.53	$-.53$	-3.8	18	-34	26	31	$-.054$
$\lvert g^a\rvert$	.048	.094	.13	.20	.67	1.2	2.5	3.9	4.8	.14
$-\langle\dot{\omega}^h\rangle$	.60	1.7	2.7	5.1	32	80	230	450	590	1.7
$\lvert g^h\rvert$	.32	.62	.86	1.3	4.5	8.2	17	26	32	.93
$\lvert\Delta\dot{\omega}\rvert$	50[a]	5[b]	4[a]	5[a]	2000[c]	100[c]	$2 \cdot 10^5$ [c]	$2 \cdot 10^5$ [c]		

Table 2: Characteristics relating to the perihelion precession rates produced by the EFE quadrupole anomaly, and the harmonic anomaly: $\kappa\lambda$ as defined in eq.(38); the predicted precession rate $\langle\dot{\omega}\rangle$ for the quadrupole anomaly (calculated assuming that the orbit is in the ecliptic–not a very good approximation for Pluto and Icarus.); the mean anomalous quadrupole acceleration on the orbit, $|g^a| = |q_{zz}|a$; the precession rate due to the harmonic anomaly $\langle\dot{\omega}^h\rangle$; its mean acceleration $|g^h| = Aa_0a/R_M$; and an indication of existing tightness of conventional fits, $|\Delta\dot{\omega}|$, that I was able to find in the literature: from (a) Pitjeva (2005), (b) Pitieva as quoted in Fienga & al. (2009), (c) Fienga & al. (2009). All precession rates in units of 10^{-4}"/c. All accelerations in units of 10^{-12}cm s^{-2}. All values are calculated for $a_0 = 10^{-8}$cm s^{-2}, $q = -0.1$, and $A = 2/3$.

Famaey (2006), Sanders & Noordermeer (2007), and Milgrom & Sanders (2008)–have shown that rotation-curve analysis can, at present, be used to constrain μ in its transition region. However, all existing studies make use of the prediction of modified-inertia formulations. These establish a universal, algebraic relation of the form (1) between the Newtonian and MOND accelerations *for circular orbits* (Milgrom 1994). For consistency, such attempts to constrain μ for use in the present context should use the predictions of the modified Poisson formulation (or a multi-potential theory, if we are dealing with one). This is not an easy task since this theory does not provide an easy-to-use formula, but would require for each galaxy model, and for each choice of parameters, a separate numerical solution of the nonlinear Poisson equation. Brada & Milgrom (1995) did show that the differences in prediction of the two formulations are not very large, but they are of the same magnitude as the differences produced by different interpolation functions.

Interesting constraints on the form of μ in the transition region will also come from studies of the dynamics perpendicular to the galactic disk at the solar position (Milgrom 1983, and see a recent study of the prospects of such analysis in Bienaymé & al. 2009).

We thus cannot, at present, predict the exact strength of the effect, for lack of exact knowledge of the interpolating function and the value of η. However, the presence of the effect can be easily identified if we can observe the anomaly in the motion of more than one planet: The effect for all planets hinges on only one parameter, q, and the dependence of the effect on orbital parameters is very distinctive. The anomalous acceleration, and the precession rate, increases with orbital radius in a prescribed way. The precession rate depends also on the orientation of the orbit relative to the galactic center in a known manner [through $\lambda(\psi_0)$], which distinguishes it from a spherical anomaly (or from an aligned, axisymmetric one). In contrast, a central quadrupole, such as is produced by an oblate sun, for example, has very different characteristics: its effect decreases sharply with increasing orbital radius, and the perihelion precession rate it produces does not depend on the orientation of the orbital major axis with respect to the quadrupole axis (our ψ_0) when that axis is in the orbital plane (or perpendicular to it).

The MOND anomalous acceleration becomes of order a_0 at radii $\sim R_M$ and beyond[19], and could have important effects on object in the outer solar system, such as the long period comets. MOND could introduce completely new insights into the study of the formation, structure, and evolution of the Oort cloud, and shed new light on possible mechanisms for directing Oort cloud object into the inner solar system. But considerations of these sort are beyond the scope of my present discussion.

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[19] At these radii, the field no more has the quadrupole form that which obtains near the sun.

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A. Properties of the phantom mass

There are several properties of the phantom mass that we can derive analytically.

A.1. The Newtonian field of the phantom mass vanishes at the position of the sun

This is a simple corollary of a more general result: If $\rho(\mathbf{r})$ is a confined density distribution of total mass M, in a constant, background (MOND) field $\mathbf{g}_g$, then the force acting on this distribution is $M\mathbf{g}_g$ (Bekenstein and Milgrom 1984). On the other hand this force is also $M\mathbf{g}_g$ plus the force due to the phantom mass. It follows then that the total anomalous force on the true mass ρ always vanishes exactly. This is a generalization to the case of an external field of the following observation: A mass distribution does not self propel in either Newtonian gravity or in the modified-Poisson formulation. This means that the difference of the net MOND force and Newtonian force on the totality of (true) mass must vanish. But this difference is just the Newtonian force of the phantom matter on the true matter, which must then vanish. The phantom matter does, of course, exert differential forces on different parts of the true mass–these are the MOND corrections to the inter-body forces in the system–but they all add up to zero. In our present configuration the force on the Sun is $M_\odot\mathbf{g}_p(0)$, where $\mathbf{g}_p(0)$ is the Newtonian field of the phantom mass at the position of the sun, and it must thus vanish. In the very near vicinity of the Sun the field of the phantom mass–i.e., the anomaly we are after–is predominantly a quadrupole.

A.2. The column density along the symmetry axis

On dimensional grounds, the phantom column density along the z axis can be written as

$$\Sigma_p(0) = \frac{a_0}{2\pi G}\gamma(\eta). \tag{A1}$$

Integrating expression (10) for ρ_p along the symmetry axis in three segments between which $\mathbf{e}$ changes sign: from $-\infty$ to the critical point, from there to the sun's position, and from there to ∞, we get

$$\gamma = \mathcal{U}(\infty) - \mathcal{U}(0) = \int_0^\infty L(x)dx \tag{A2}$$

$[L(0) = 1,\ L(\infty) = 0]$. The dimensionless factor γ is thus independent of η and depends only on the choice of μ. For the limiting form of $\mu(x)$ we have $\mathcal{U}(\infty) - \mathcal{U}(0) = 1$, hence in this case the

column density is exactly $a_0/2\pi G$. For μ_2 we have $\gamma = \pi/2$. Note also that γ diverges if $\delta\mu(x)$ behaves at large x as x^{-1} or slower.

A.3. The phantom density takes up both signs near the critical point

At the critical point $|\vec{\nabla}\phi|$ vanishes, so it must be increasing in all directions emanating thence. Take any surface, Σ, of constant $|\vec{\nabla}\phi|$ that surrounds the critical point, but excludes the sun. Applying Gauss's theorem to the modified Poisson eq.(2) for the volume enclosed by Σ, which is devoid of (true) mass, gives $\int_\Sigma \vec{\nabla}\phi \cdot \mathbf{d}\sigma = 0$. Thus, since $\vec{\nabla}|\vec{\nabla}\phi|$ is parallel to $\mathbf{d}\sigma$ everywhere on the surface, $\vec{\nabla}|\vec{\nabla}\phi| \cdot \vec{\nabla}\phi$ must take up both signs on every such surface. Expression (9) for ρ_p then tells us that so must ρ_p.

A.4. The total mass

The total phantom mass is $M_p = M_\odot \beta(\eta)$. The dimensionless factor, β, can be found by applying Gauss theorem to eq.(8) in a spherical volume whose surface S tends to infinity.

$$M_p = \frac{1}{4\pi G}\int_S (1-\mu)\vec{\nabla}\phi \cdot \mathbf{d}\sigma. \tag{A3}$$

Here $\vec{\nabla}\phi$ is the MOND field; its asymptotic form is known analytically and given in eq.(32) of Bekenstein & Milgrom (1984). Using this expression one finds

$$\beta(\eta) = \mu_g^{-1} L_g^{-1/2} \sin^{-1}\left(\frac{L_g}{1+L_g}\right)^{1/2} - 1, \tag{A4}$$

where $L_g = L(\eta)$, $\mu_g = \mu(\eta)$.

With the limiting form of μ, and $\eta > 1$, we have $\mu_g = 1$ and $L_g = 0$; so the total phantom mass vanishes (but not ρ_p, of course). It is easy to see this directly. In this case $\mathcal{P}$ is bounded, and excludes the sun. Outside its boundary $\mu = 1$. So, on any boundary, Σ surrounding $\mathcal{P}$ we have $\mu\vec{\nabla}\phi = \vec{\nabla}\phi$. The surface integral of $\mu\vec{\nabla}\phi$ vanishes, if Σ excludes the sun, as it is proportional to the true mass in $\mathcal{P}$, and this means that $\int_\Sigma \vec{\nabla}\phi \cdot \mathbf{d}\sigma = 4\pi G M_p$ also vanishes.

If μ does not have the limiting form, M_p is finite. For example for $\mu = \mu_2 = x/(1+x^2)^{1/2}$, we have for $\eta = 2$, $\beta \approx 0.05$, and for $\eta = 1$, $\beta \approx 0.23$. The phantom mass is always non-vanishing if $\mu(\eta) < 1$. But note that M_p in itself is not an indication for the strength of the anomalous quadrupole near the sun. Taking an extreme case, when $\eta \to 0$, we have $M_p \to \infty$ (reflecting the fact that it then constitutes an asymptotically isothermal phantom halo around the central mass). But the quadrupole anomaly actually goes to 0 in this case as the phantom mass forms a spherical cavity around the sun.

All the above applies to a single star in a constant field. In reality, the phantom masses of a star cuts off as it runs into those of other stars. It is easy to see that the phantom mass of a galaxy is just the totality of phantom masses of individual constituents; e.g., by applying Gauss theorem to a large volume written as the union of its sub volumes. One can then mimic the effects of MOND with a smoothly distributed dark matter, but only when the dynamics is probed with very course graining such as in rotation curve analysis. Unlike the hypothetical CDM, the MOND phantom matter is highly granular on the scales of distances between object in the galaxy, with regions of negative density, etc.. This, including the EFE we discuss in this paper, are not mimicked by CDM.

A.5. Space integrals weighted by ρ_p

One encounters various quantities of the form

$$F = \int \rho_p(\mathbf{r}) f(\mathbf{r}) d^3\mathbf{r}, \tag{A5}$$

where $f(\mathbf{r})$ can be any tensorial function of $\mathbf{r}$. Examples are the various multipoles of the phantom mass, and its Newtonian potential and acceleration fields. Here I derive useful expressions for such quantities writing them as integrals over $\vec{\nabla}\phi$ itself, instead of ρ_p, which involves second derivatives of ϕ. These are quite useful for numerical evaluations.

Using expression (8) for ρ_p we can write

$$F = \frac{1}{4\pi G} \int f(\mathbf{r}) \vec{\nabla} \cdot [(1-\mu)\vec{\nabla}\phi] d^3\mathbf{r}. \tag{A6}$$

Integrating by parts and applying Gauss's theorem

$$F = \frac{1}{4\pi G} \int f(\mathbf{r})(1-\mu)\vec{\nabla}\phi \cdot \mathbf{d}\sigma - \frac{1}{4\pi G} \int (1-\mu)(\vec{\nabla}\phi \cdot \vec{\nabla}) f(\mathbf{r}) d^3\mathbf{r}. \tag{A7}$$

We have already had one example of such an expression where F is the total phantom mass, with $f = 1$. The second integral then vanishes, and the surface integral at infinity can be evaluated for the case of a finite external field (for $\mathbf{g}_g = 0$ the surface integral diverges). In general, we do not have a prescription for calculating the surface integral. However, there are cases when we can evaluate it. For example, if μ takes its limiting form, and if $\eta > 1$, the surface integral vanishes identically over any surface surrounding $\mathcal{P}$, and we can write

$$F = \frac{1}{4\pi G} \int_{\mathcal{P}} (\mu - 1)(\vec{\nabla}\phi \cdot \vec{\nabla}) f(\mathbf{r}) d^3\mathbf{r}. \tag{A8}$$

In this case we get for the dipole moment, with $f = \mathbf{r}$

$$\mathbf{D} = \frac{1}{4\pi G} \int_{\mathcal{P}} (\mu - 1)\vec{\nabla}\phi d^3\mathbf{r}. \tag{A9}$$

For the quadrupole we have from eq.(A8), with $f = -(r^2\delta_{ij} - 3r_ir_j)/2$,

$$Q_{ij} = -\frac{1}{4\pi G}\int_{\mathcal{P}}(\mu - 1)[\mathbf{r}\cdot\vec{\nabla}\phi\delta_{ij} - \frac{3}{2}(\phi_{,i}r_j + \phi_{,j}r_i)]d^3\mathbf{r}, \tag{A10}$$

where $\mathbf{e}$ is a unit vector in the direction of $\vec{\nabla}\phi$. In particular

$$Q_{zz} = -\frac{1}{4\pi G}\int_{\mathcal{P}}(\mu - 1)(\mathbf{r}\cdot\vec{\nabla}\phi - 3z\phi_{,z})d^3\mathbf{r}. \tag{A11}$$

The anomaly itself given in eq.(11), corresponds to $f = G|\mathbf{r} - \mathbf{R}|^{-3}(\mathbf{r} - \mathbf{R})$. In this case, because of $f(\mathbf{r}) \to 0$ at infinity, we can evaluate the surface integral in eq.(A7). We have for any choice of $\mu(x)$ compatible with MOND, and for any η

$$\mathbf{g}^a(\mathbf{R}) = \frac{1}{4\pi}\int\frac{(\mu - 1)}{|\mathbf{r} - \mathbf{R}|^3}[\vec{\nabla}\phi - 3(\vec{\nabla}\phi\cdot\mathbf{n})\mathbf{n}]d^3\mathbf{r} - \frac{1}{3}(1 - \mu_g)\mathbf{g}_g, \tag{A12}$$

where $\mathbf{n}$ is a unit vector in the direction of $\mathbf{r} - \mathbf{R}$.

Another version, which might also be useful, is gotten when writing expression (8) for ρ_p as

$$\rho_p = \frac{1}{4\pi G}\vec{\nabla}\cdot[\vec{\nabla}\phi - (\mu/\mu_g)\vec{\nabla}\phi] + [(1/\mu_g) - 1]\rho. \tag{A13}$$

This now gives

$$F = (1/\mu_g - 1)F^\rho + \frac{1}{4\pi G}\int(\mu/\mu_g - 1)(\vec{\nabla}\phi\cdot\vec{\nabla})f(\mathbf{r})d^3\mathbf{r} - \frac{1}{4\pi G}\int(\mu/\mu_g - 1)f(\mathbf{r})\vec{\nabla}\phi\cdot\mathbf{d}\sigma, \tag{A14}$$

where F^ρ is the quantity F calculated with the true mass, and the convergence of the surface integral has now been improved.

A.6. Asymptotic behavior of ρ_p

The asymptotic behavior of ρ_p can be derived from the known behavior of the MOND field there. As derived in Bekenstein & Milgrom (1984), the internal potential is, asymptotically (for $u \gg \eta^{-1/2}R_M$),

$$\phi_{in} \approx -\frac{MG}{\mu_g(1 + L_g)^{1/2}}\left(x^2 + y^2 + \frac{z^2}{1 + L_g}\right)^{-1/2}. \tag{A15}$$

The asymptotic acceleration field is thus

$$\mathbf{g}_{in} \approx -\frac{MG}{\mu_g(1 + L_g)^{1/2}}\left(x^2 + y^2 + \frac{z^2}{1 + L_g}\right)^{-3/2}[x, y, \frac{z}{1 + L_g}], \tag{A16}$$

and the phantom density

$$\rho_p \approx \frac{-M}{4\pi\mu_g}\frac{L_g}{(1 + L_g)^{3/2}}\left(x^2 + y^2 + \frac{z^2}{1 + L_g}\right)^{-5/2}\left(x^2 + y^2 - \frac{2z^2}{1 + L_g}\right) =$$

$$= -\frac{M}{4\pi\mu_g}\frac{L_g}{(1+L_g)^{3/2}}u'^{-3}(1-3z'^2/u'^2), \tag{A17}$$

where $x' = x$, $y' = y$, $z' = z(1+L_g)^{-1/2}z$, $u' = (x'^2 + y'^2 + z'^2)^{1/2}$. We see that ρ_p decreases asymptotically as u^{-3}.

The z component of the anomalous acceleration on the z axis and the x component on the x axis are respectively

$$g^a_z(z) \approx -\frac{MG}{z^2}\left[\frac{1}{\mu_g} - 1\right], \quad g^a_x(x) \approx -\frac{MG}{x^2}\left[\frac{1}{\mu_g(1+L_g)^{1/2}} - 1\right] \tag{A18}$$

We learn from this that at the same distance along the two axes, the MOND acceleration on the symmetry axis is larger by a factor $(1+L_g)^{1/2}$ than that in the perpendicular direction: there is a stronger pull towards the Sun along the z axis. This is then also the case for the anomalous acceleration, since the Newtonian one is spherical. In fact, we see that the anomaly along the z axis is always attractive towards the sun, asymptotically. This is the opposite to what a negative value of q gives near the sun.

B. Properties of the surrogate phantom mass

B.1. Its Newtonian field vanishes at the position of the sun

Here I show that the Newtonian field of the surrogate, phantom matter vanishes at the position of the sun. I first show, in general, that the force of a true density ρ on its surrogate, phantom mass vanishes identically. This force is

$$\tilde{\mathbf{F}}^p = -\int \tilde{\rho}_p \vec{\nabla}\phi^\rho d^3\mathbf{r} = \frac{1}{4\pi G}\int (\vec{\nabla}\cdot\mathbf{g}^* + 4\pi G\rho)\vec{\nabla}\phi^\rho d^3\mathbf{r}, \tag{B1}$$

where $\vec{\nabla}\phi^\rho$ is the Newtonian field of ρ alone, and the integral is over all space. The second term vanishes, because it constitutes the Newtonian force of ρ on itself. The ith component of $\tilde{\mathbf{F}}^p$ can then be written as

$$\tilde{F}^p_i = \frac{1}{4\pi G}\int \vec{\nabla}\cdot\mathbf{g}^*\phi^\rho_{,i}d^3\mathbf{r} = -\frac{1}{4\pi G}\int (\nu\phi^N_{,k})_{,k}\phi^\rho_{,i}d^3\mathbf{r} = -\frac{1}{4\pi G}\int [(\nu\phi^N_{,k}\phi^\rho_{,i})_{,k} - \nu\phi^N_{,k}\phi^\rho_{,i,k}]d^3\mathbf{r}. \tag{B2}$$

Here, ϕ^N is the potential of the full Newtonian field–that of ρ plus the background field: $\mathbf{g}_N = -\vec{\nabla}\phi^N$, and ν is a function of $|\mathbf{g}_N|/a_0$. (A subscript $,i$ is the derivative with respect to the ith coordinate.) In the second term in the integrand we can replace $\phi^\rho_{,i,k}$ with $\phi^N_{,i,k}$, because $\phi^\rho_{,i}$ and $\phi^N_{,i}$ differ by the constant background acceleration. We then write this term, now in the form $\nu\phi^N_{,k}\phi^N_{,k,i} = N_{,i}[(\vec{\nabla}\phi^N/a_0)^2]$, where $N'(z) \equiv \nu(\sqrt{z})/2$. Applying Gauss theorem we then have

$$\tilde{\mathbf{F}}^p = -\frac{1}{4\pi G}\int (\nu\vec{\nabla}\phi^\rho\vec{\nabla}\phi^N\cdot\mathbf{d}\sigma - N\mathbf{d}\sigma), \tag{B3}$$

where the integration is over the surface at infinity. In the first term $\vec{\nabla}\phi^\rho$ decreases as r^{-2}; so the other factors can be taken at their constant limiting value at infinity: ν becomes $\nu_g = \nu(\eta_N)$. In the second term, we take the first order expansion $N[(\vec{\nabla}\phi^N/a_0)^2] \rightarrow N(\eta_N^2) - \nu_g \vec{\nabla}\phi^\rho \cdot \mathbf{g}_g^N$. The constant, limiting value of N gives a vanishing integral, and we thus have

$$\tilde{\mathbf{F}}^p = \frac{\nu_g}{4\pi G}\int(\vec{\nabla}\phi^\rho \mathbf{g}_g^N \cdot \mathbf{d}\sigma - \vec{\nabla}\phi^\rho \cdot \mathbf{g}_g^N \mathbf{d}\sigma). \tag{B4}$$

Taking the z axis in the direction of $-\mathbf{g}_g^N$, we have

$$\tilde{\mathbf{F}}^p = -\frac{\nu_g g_g^N}{4\pi G}\int(\vec{\nabla}\phi^\rho d\sigma_z - \phi^\rho_{,z}\mathbf{d}\sigma). \tag{B5}$$

By applying Gauss's theorem in reverse, writing the right hand side as a volume integral, we see that the two terms cancel, and thus $\tilde{\mathbf{F}}^p = 0$. This also means that the net force of the phantom matter on the true matter always vanishes. For our special case, it follows that the Newtonian force of $\tilde{\rho}_p$ on the Sun vanishes. However, since the Sun is a point mass, it follows that the Newtonian acceleration field of $\tilde{\rho}_p$ vanishes at the position of the sun.

B.2. Its column density along the symmetry axis

It is easy to gather that $\tilde{\rho}_p$ can be gotten from equations exactly like eqs.(9)(10), with $\vec{\nabla}\phi$ there replaced by $-\mathbf{g}^*$, with the same $\mathcal{U}$ function. Its column density along the symmetry axis is thus identical to that of ρ_p, and equals $a_0 G^{-1}\gamma$ with $\gamma = (2\pi)^{-1}\int_0^\infty L(x)$.

B.3. it takes up both signs

As in the case of ρ_p, $\tilde{\rho}_p \propto \vec{\nabla}|\mathbf{g}_N| \cdot \mathbf{g}_N$, and for similar reasons this takes up both signs on any surface of constant $|\mathbf{g}_N|$ surrounding the critical point, and excluding true matter. This can also be seen explicitly in expression (C3) for $\hat{\rho}_p$, which is a good approximation for $\tilde{\rho}_p$ around the critical point.

B.4. The total mass

Use eq.(20) for $\tilde{\rho}_p$ to write

$$\tilde{\rho}_p = -\frac{1}{4\pi G}\vec{\nabla}\cdot[(\nu - 1)\mathbf{g}_N]. \tag{B6}$$

Applying Gauss's theorem to eq.(20) we have

$$\tilde{M}_p = -\frac{1}{4\pi G}\int(\nu - 1)\mathbf{g}_N \cdot \mathbf{d}\sigma, \tag{B7}$$

on the surface at infinity, where we can write $\mathbf{g}_N \rightarrow \mathbf{g}_g^N + \mathbf{g}^\rho$. Here $\mathbf{g}_g^N$ is the Newtonian external field, and $\mathbf{g}^\rho = -MG\mathbf{u}u^{-3}$ is the asymptotic Newtonian field of the true mass alone. Expanding around $\mathbf{g}_g^N$, we get

$$\tilde{M}_p/M = \nu(\eta_N) - 1 + \eta_N \nu'(\eta_N)/3, \tag{B8}$$

where $\eta_N = \mu(\eta)\eta$ is the Newtonian galactic field. Using the relations $\nu(\eta_N) = 1/\mu(\eta)$ and $\nu'(\eta_N) = -\eta^{-1}\mu(\eta)^{-2}L(\eta)[1+L(\eta)]^{-1}$ we have

$$\tilde{M}_p/M = \frac{3+2L_g}{3\mu_g(1+L_g)} - 1. \tag{B9}$$

Since $0 \le L_g \le 1$ this gives very similar values to those of the exact expression (A4). In the MOND regime $L_g \approx 1$, $\mu_g \ll 1$ it gives $5/6\mu$ compared with $\pi/4\mu$ for M_p.

B.5. integrals weighted by $\tilde{\rho}_p$

We shall also need similar expression to eq.(A7) when $\tilde{\rho}_p$ is used instead of ρ_p. Repeating the arguments in A.5 using expression (B6), we see that we get an expression similar to eq.(A7) with $\vec{\nabla}\phi$ replaced by $-\mathbf{g}^*$. For example, the surrogate anomaly is

$$\tilde{\mathbf{g}}^a(\mathbf{R}) = \frac{1}{4\pi}\int \frac{(\nu-1)}{|\mathbf{r}-\mathbf{R}|^3}[\mathbf{g}_N - 3(\mathbf{g}_N\cdot\mathbf{n})\mathbf{n}]d^3\mathbf{r} - \frac{1}{3}(\nu_g - 1)\mathbf{g}_g^N. \tag{B10}$$

The surrogate quadrupole moment for the limiting form of μ, which I shall need below, is

$$\tilde{Q}_{zz} = -\frac{1}{4\pi G}\int (\nu-1)(\mathbf{r}\cdot\mathbf{g}_N - 3zg_z^N)d^3\mathbf{r}. \tag{B11}$$

B.6. The asymptotic behavior of $\tilde{\rho}_p$

Using the algebraic relation (1) it is readily seen that the asymptotic behavior of $\mathbf{g}^*$ is

$$\mathbf{g}^* - \mathbf{g}_g \approx -\frac{MG}{\mu_g u^3}\left(x, y, \frac{z}{1+L_g}\right). \tag{B12}$$

The asymptotic behavior of $\tilde{\rho}_p$ derived from eq.(B12) is

$$\tilde{\rho}_p \approx -\frac{M}{4\pi\mu_g}\frac{L_g}{1+L_g}u^{-3}(1-3z^2/u^2) \tag{B13}$$

$[u = (x^2+y^2+z^2)^{1/2}]$. We see a similar normalization and radial dependence as in ρ_p, but a different angular dependence.

B.7. Delineation of $\tilde{\mathcal{P}}$ for the limiting μ and $\eta_N > 1$

The region $\tilde{\mathcal{P}}$ is defined by $|\mathbf{g}^*| < a_0$, or equivalently by $|\mathbf{g}_N| < a_0$. The Newtonian acceleration at position $\mathbf{u}$ relative to the Sun is given in eq.(21). The ray from the Sun making an angle θ with the z axis cuts $\tilde{\mathcal{P}}$ at

$$u_z^{\pm} = \left[\frac{\eta_N \cos\theta \pm (1 - \eta_N^2 \sin^2\theta)^{1/2}}{(\eta_N^2 - 1)\cos^{-2}\theta}\right]^{1/2}, \qquad \text{(B14)}$$

which give the boundaries of $\tilde{\mathcal{P}}$. The viewing angle of $\tilde{\mathcal{P}}$ from the Sun occurs where $u_z^+ = u_z^-$, for which $\sin\theta = 1/\eta_N$. $\tilde{\mathcal{P}}$ intersects the z axis at $u_z^{\pm} = (\eta_N \mp 1)^{-1/2}$. The critical point is at $\eta_N^{-1/2}$ (all in units of R_M). Thus, the linear size of $\tilde{\mathcal{P}}$ decreases as $\eta_N^{-3/2}$ for large η_N, while the distance of $\tilde{\mathcal{P}}$ from the Sun decreases only as $\eta_N^{-1/2}$. This will become important below.

For the special case $\eta_N = 1$, $\tilde{\mathcal{P}}$ extends everywhere to the positive z side of the surface $z_-(r) = \{[(r^4 + 2)^{1/2} - r^2]/2\}^{1/2}$, with the critical point at $z = 1$.

C. Further approximations for the limiting form of μ

Here I derive an approximate, closed expression for the dimensionless, surrogate anomaly $\tilde{q}(\eta)$ for the limiting form of μ. It can be used to check numerical evaluations of the anomaly, and it is also a closed expression for some sort of a lower limit on the anomaly for $\eta > 1$. The approximation is based on the observation that the linear size of $\tilde{\mathcal{P}}$ becomes increasingly small relative to its distance from the sun, as η_N increases (see B.7). In this limit we can use for $\mathbf{g}_N$ in $\tilde{\mathcal{P}}$ its linear expansion around the critical point, where $\mathbf{g}_N$ vanishes. This means writing

$$\mathbf{g}_N \approx -\frac{\eta_N^{3/2} a_0}{R_M}(\mathbf{r} - 3z\mathbf{e}_z), \qquad \text{(C1)}$$

instead of the exact expression (21) ($\mathbf{r}$ is now measured from the critical point, unlike $\mathbf{u}$, which is measured from the sun). With this approximation, the region $\tilde{\mathcal{P}}$ becomes the interior, $\hat{\mathcal{P}}$, of the ellipsoid

$$x^2 + y^2 + 4z^2 = \ell^2, \qquad \text{(C2)}$$

centered at the critical point, where $\ell = \eta_N^{-3/2} R_M$.

The phantom density in this approximation, $\hat{\rho}_p$, is given by

$$\hat{\rho}_p = -\frac{1}{6}\rho_\odot \eta_N^{3/4} \frac{x^2 + y^2 - 8z^2}{(x^2 + y^2 + 4z^2)^{5/4}}, \qquad \text{(C3)}$$

where $\rho_\odot = 3M_\odot/4\pi R_M^3$; it is axisymmetric, but also symmetric under reflection in z. This means that, unlike ρ_p, and $\tilde{\rho}_p$, it does not have a dipole moment. One can check that the Newtonian field of $\hat{\rho}_p$ does not vanish on the z axis, except at the origin. In particular, in itself, it does not vanish at the sun's position, as required. We cannot then use expression (C1) directly in eq.(23) to get

the desired approximate expression. In replacing $\tilde{\rho}_p$ by $\hat{\rho}_p$, the quadrupole moment of the phantom mass, Q_{ij} (not to be confused with the quadrupole anomaly), is expected to be well approximated, but we have completely lost the important contribution of its dipole moment $\mathbf{D}$, which is needed to balance the quadrupole and annihilate the field at the sun's position. The dipole moment derives from the small z asymmetry that does not affect Q_{ij} much. I thus use the above approximation only for calculating the quadrupole moment. I imagine that $\hat{\rho}_p$ is slightly distorted so as to attain a dipole moment. If we know the quadrupole moment we can determine the dipole moment by requiring that they cancel at the sun, use this to eliminate the dipole, and express the whole effect in terms of the quadrupole. To recapitulate, I approximate the anomaly–the Newtonian field of the phantom mass-by its dipole-plus-quadrupole contribution, writing for the potential[20]

$$\hat{\phi}^a(\mathbf{R}) = -\frac{G}{R^3}\mathbf{D}\cdot\mathbf{R} - \frac{G}{R^5}R^iR^jQ_{ij}. \tag{C4}$$

($\mathbf{R}$, like $\mathbf{r}$, is measured from the critical point.) Now, to have the acceleration field of $\hat{\phi}^a$ vanish at the sun's position we have to have[21]

$$\mathbf{D} = \frac{-3}{2\tilde{R}_0}Q_{zz}\mathbf{e}_z. \tag{C5}$$

(Here, $\tilde{R}_0$ is the distance from the Sun to the critical point of $\tilde{\mathcal{P}}$; $\tilde{R}_0 = \eta_N^{-1/2}R_M$.) I put this value of $\mathbf{D}$ back into the expression for $\hat{\phi}^a$ to get the desired anomaly in this approximation, which requires only knowledge of the quadrupole moment. From this we can identify the anomalous quadrupole and find

$$\hat{q}^a_{xx} = \hat{q}^a_{yy} = -\frac{1}{2}\hat{q}^a_{zz} = \frac{3G}{2R_0^5}Q_{zz} \equiv -\frac{a_0}{2R_M}\hat{q}(\eta). \tag{C6}$$

Using expression (C1) for $\mathbf{g}_N$ in eq.(B11)–the formula for the quadrupole moment–then gives straightforwardly

$$Q_{zz} \approx \frac{a_0R_M^4}{90\eta_N^6G}. \tag{C7}$$

The scaling with this particular power of η_N is understood as follows: For large η_N, the linear size, d, of $\tilde{\mathcal{P}}$ is $\propto R_M\eta_N^{-3/2}$ in all directions . The characteristic surface column density is always $\Sigma \propto a_0/G$; the quadrupole moment, which scales as Σd^4 must then be $Q \propto a_0G^{-1}R_M^4\eta_N^{-6}$. This gives finally

$$\hat{q}(\eta) = -\eta_N^{-7/2}/30. \tag{C8}$$

Note that our expressions (C7)(C8), and the approximation behind them, must break down for values of η slightly above 1, or lower. We know, for example, that the anomaly should vanish in the limit $\eta_N \to 0$, which relation (C8) fails completely to account for.

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[20] $\mathbf{D} = \int \rho_p(\mathbf{r})\mathbf{r}d^3\mathbf{r}$, and $Q_{ij} = -\frac{1}{2}\int \rho_p(\mathbf{r})(r^2\delta_{ij} - 3r_ir_j)d^3\mathbf{r}$. Q_{ij} is a traceless matrix, and with the symmetry of the problem it is diagonal, with $Q_{xx} = Q_{yy} = -Q_{zz}/2$.

[21] This relation does not need to hold exactly since higher multipoles may also contribute to the field value at the sun, but it should hold to the same accuracy as the dipole-plus-quadrupole approximation.

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